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13. ABSTRACT (Maximum 200 words)
Trade-offs between stability margin and performance are considered in two and three-degree-of-freedom multivariable control systems using a Wiener-Hopf design approach. Maximum improvement in an approximate measure of stability margin is achieved at the expense of a prescribed increase in the quadratic cost functional measuring system performance. In order to attain an analytical solution to this fundamental trade-off problem, the approximate measure of stability margin chosen is also a quadratic cost function. A novel approach is introduced which allows structured perturbations in the coprime polynomial matrix fraction description of the plant transfer matrix to be taken into account. As a consequence, it is believed that the use of an approximate measure of stability margin is mitigated. Moreover, if needed, the solution obtained could serve as a very good initial one from which to search for better solutions iteratively. The aforementioned control design methodology was implemented on the available testbeds for advanced weapon pointing systems at the Picatinny Army Arsenal in New Jersey.

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Final Report

Robust Wiener-Hopf Design for Multivariable Control Systems and Applications to Vibration Suppression on a Weapon Pointing System

by

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Summary of the Effort

Trade-offs between stability margin and performance are considered in two and three-degree-of-freedom multivariable control systems using a Wiener-Hopf design approach. Maximum improvement in an approximate measure of stability margin is achieved at the expense of a prescribed increase in the quadratic cost functional measuring system performance. In order to attain an analytical solution to this fundamental trade-off problem, the approximate measure of stability margin chosen is also a quadratic cost function. A novel approach is introduced which allows structured perturbations in the coprime polynomial matrix fraction description of the plant transfer matrix to be taken into account. As a consequence, it is believed that the use of an approximate measure of stability margin is mitigated. Moreover, if needed, the solution obtained could serve as a very good initial one from which to search for better solutions iteratively. The aforementioned control design methodology was implemented on the available testbeds for advanced weapon pointing systems at the Picatinny Army Arsenal in New Jersey.

Keywords: Wiener-Hopf Design, Stability Margin, Performance, Multivariable Linear Systems.

Contents

1. Introduction

The feedback control system considered is shown in Fig. 1.1. This system is referred to as a 3DOF feedback system because of the three real rational transfer matrices $C_u(s)$, $C_w(s)$, and $C_z(s)$ which can be chosen in the design of the controller. When no feedforward transducer is available, $C_z(s) = 0$ and the feedback system reduces to a 2DOF system. Optimal and suboptimal designs for the system shown in Fig. 1.1 are treated in [1] and form the basis for the work presented here. In fact, the notation used here is consistent with that used in [1] for easy reference. The transfer matrices given in the blocks of Fig. 1.1 are real and rational. The matrices $F_t(s)$ and $T_0(s)$ are n $\times$ n, $P(s)$ is n$\times$m, $L_t(s)$ is $l \times r$, and $P_0(s)$ is $n \times r$. The dimensions of all other vectors and matrices are taken to be compatible with these.

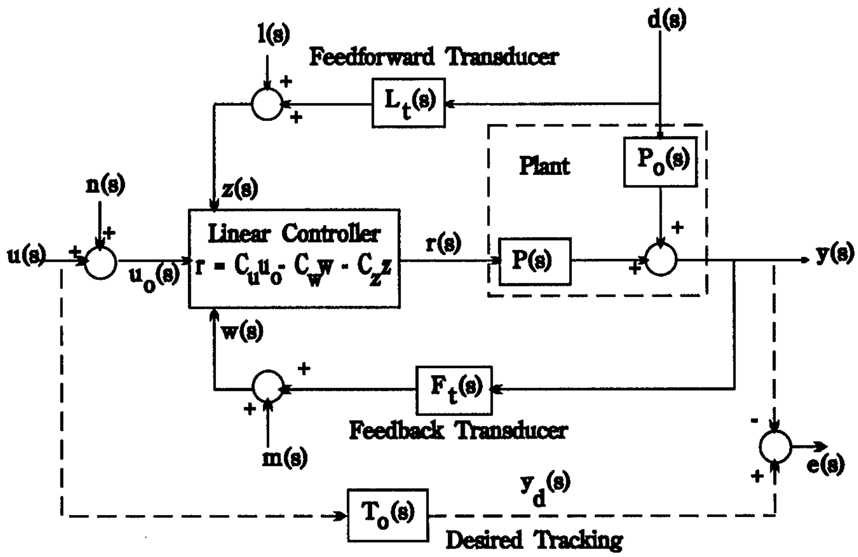

Figure 1.1: The 3DOF system.

The Wiener-Hopf design philosophy for two and three-degree-of-freedom (2DOF and 3DOF) multivariable control systems is described in [1–3]. In these works, a quadratic cost functional is postulated to measure system performance. The cost functional takes into account such issues as tracking error, saturation, and sensitivity to small parameter variations. The main contribution of this earlier work is the identification of the subset of all stabilizing controllers *for which the cost functional is finite.* This subset is parameterized for two-degree-of-freedom (2DOF) systems in terms of two real rational matrix parameters each of which is strictly proper and analytic in the closed right hand side of the complex plane. An identical result holds for three-degree-of-freedom (3DOF) systems except that the parameterization is in terms of three rather than two such matrices. Since the 2DOF system can be treated as a special case of the 3DOF system [1], the ideas here are first presented and developed for the 3DOF case. The results for the 2DOF case are then easily derived.

The cost functionals are minimized in both the 2DOF and 3DOF cases when the matrix parameters are chosen to be null matrices. When non-null choices are made for any of the matrix parameters, an increase in the cost functional results which is related to the matrix parameters through simple formulas. Similar results are available in [4], [5] for the so-called generalized plant configuration. In this configuration, all system components other than the controller are represented by the generalized plant. This point of view is especially appealing to those who wish to concentrate on solving the mathematical problems associated with controller design. Their results are then applicable to all possible feedback system configurations. On the other hand, it is also useful for engineers to have design formulas available for a specific configuration whose structure has wide application. Such a structure is the one employed in the 3DOF system. Indeed, it is hard to imagine any control system applications which are not included in the 3DOF case, although a 4DOF system involving the generation of a signal to be used for diagnostic purposes is described in [6]. For the 3DOF structure, the relationship of the matrix design parameters to important engineering issues can be exploited. In particular, those matrix parameters which are connected to system robustness can be used to trade-off optimal nominal system performance against improved robustness with respect to plant uncertainty. A straightforward methodology for achieving this trade-off is described in this paper.

The direct nature of the methodology employed here together with the careful attention paid to the increase in the value of the original quadratic cost functional contrasts with the design point of view employed in the loop transfer recovery (LTR) methodology [7–14]. Specifically, the cost functional is viewed here as a truly indicative measure of nominal system performance and not one whose weights can be manipulated to achieve desired engineering goals. Instead, the matrix parameters are viewed as the objects to be manipulated and the cost increments associated with any such manipulation is kept to the forefront. In addition, a one-step process instead of a two-step process is used to find the unique choice for the matrix parameters. This unique choice is the solution of a constrained optimization problem. That is, one in which a quadratic or H_2 measure of the stability margin is optimized for a prescribed increment in the cost functional. Moreover, no special considerations are needed when the plant transfer matrix is nonminimum phase and/or sensor noise as well as disturbance inputs are present. Also the design procedure begins with coprime polynomial matrix fraction descriptions for the plant and sensors and can be carried out without state variable models for these system components.

Justification for the point of view adopted here is based on the following observations. Foremost in the design of advanced control systems is the need to satisfy performance specifications. In fact, one often sacrifices stability margin or accepts conditionally stable systems in order to realize superior performance. So one should begin with an optimal H_2 solution. When this solution exceeds specifications, the cost functional can be increased and the increment traded for an improvement in stability margin. That is, one should maximize stability margin subject to a constraint on the H_2 norm cost increment. When the stability margin obtained in this way is not adequate, then one must use a different controller for different operating profiles of the plant: typically, one should not give up performance just for the sake of operating with a single fixed controller when high performance is required.

In a recent paper [15], stability margin is addressed in a quadratic cost setting by including an additional term that accounts for stability margin in the minimized cost functional. This approach, however, was not fully explored in [15] and does not provide the perspective gained here. Here, instead, suboptimal performance is linked to stability margin through some non-null choices of the matrix parameters. In this way, the trade-off being made becomes transparent and important design formulas are obtained.

The use of a quadratic cost functional as an indicator of control system performance for systems subjected to both stochastic and shape-deterministic inputs has been a long accepted practice in the control field and needs no justification here [16,17]. Stability margin, on the other hand, is more correctly measured by an H_∞ norm [18]. So the real problem involves the application of an H_2 norm on an indicator of system performance and the application of an H_∞ norm on an indicator of stability margin, with the indicators in each case being different functions of the same matrix parameter. That is, the real problem to be solved is a mixed H_2/H_∞ problem [19–29]. The approaches in the literature for the resolution of this type of problem have typically focused on minimizing, subject to an H_∞ norm constraint, a H_2 norm. Except for [25], analytical solutions are not available. In [25], however, the generalized plant configuration is considered with state feedback. This case is impractical since not only is the state of the original physical plant required, but also the states of all the sensors. Moreover, it is argued below that when the H_∞ norm is used to measure stability margin, it is the reverse problem that is more appropriate. That is, one should instead minimize the H_∞ norm subject to a constraint on the H_2 norm. So it is believed that the preliminary efforts in [30] to trade-off excess stability margin in the form identified in [18] in order to minimize a quadratic performance measure is not headed in the right direction.

Unfortunately, minimizing the H_∞ norm subject to a constraint on the H_2 norm is also a difficult problem for which analytical solutions are not expected. In this paper, the difficulties are circumvented by choosing a meaningful, although nonrigorous, quadratic measure rather than an H_∞ measure of stability margin. This is in keeping with the use of a quadratic measure for saturation which is also approximate. In addition, in contrast with [18] where coprime rational matrix fraction descriptions for the plant are used, here polynomial matrix fractions are used and uncertainty is modelled as perturbations in the coefficient matrices of these coprime polynomial matrices. It is believed that this is a more natural way to model structural uncertainty in practice.

By working with an H_2 norm for both measures of performance and stability margin, a key result in [31] can be employed and one obtains a frequency-domain analytical solution for the design problem to within the choice of a single scalar Lagrangian multiplier. It is anticipated that this approach will often lead to satisfactory designs or at least good initial designs from which refinements can be explored. Indeed, it is possible to view the design problem treated here as a trade-off among Pareto optimal solutions for a multicriterion optimization problem [32]. Certainly, the robustness analysis tools described in [33] would also play a major role in the evaluation and modification of the control systems synthesized using the methodology presented here.

There is a danger in using an H_2 norm instead of an H_∞ norm for stability margin. In [34], for a class of scalar systems, it is established that the H_∞ norm as an indicator of system performance can be substantially larger than its minimum value when the free matrix parameter is chosen to be the one which minimizes the H_2 norm of this same indicator. Since the minimum H_∞ norm solutions are typically overly conservative [35,36] when uncertainties are structured, and since the properties of the function whose H_∞ norm was investigated in [34] differ somewhat from the properties of the function used here as an indicator of stability margin, it is expected that the potential danger of working with an H_2 norm is mitigated.

The H_2 norm trade-off problem of interest here can also be approached numerically using convex programming along the lines taken in [29,32,37,38] or indirectly through linear matrix inequalities (LMI) as described in [39,40]. The former involves approximating the infinite-dimensional set for the real, rational, strictly-proper matrix design parameter with a sequence of larger and larger finite-dimensional subsets. The latter uses convex programming to solve the LMI associated with an optimal fixed order controller. So to approximate the infinite-dimensional problem, a sequence of controllers of increasing order would have to be studied. Clearly, the analytical result given in the sequel serves as an important complement to these numerical optimization approaches. This analytical result also provides a guide for choosing a good starting point when these numerical approaches are used to search for the solution to the underlying mixed H_2/H_∞ problem, should this step be necessary.

All results in this paper are developed by working exclusively in the complex s-plane and all matrix functions of $s = \sigma + j\omega$ are assumed to be real and rational. The real part of s is denoted by Re s. For any matrix G(s), the notations G$'$, $\overline{G}$, G*, det G, and TrG are used for the transpose, conjugate, conjugate transpose, determinant, and trace of $G(s)$, respectively. The matrix $G_*(s)$ is the conjugate transpose of $G(-\overline{s})$ or $G_*(s) = G^*(-\overline{s})$ which, for real rational matrices, reduces to $G_*(s) = G'(-s)$. A real matrix $G(s)$ is called para-Hermitian when $G = G_*$. A diagonal matrix G with g_i in the i-row, i-column, i=1 $\rightarrow$ n, is denoted by G=diag$\{g_1, g_2..., g_n\}$. The Kronecker product of two matrices is denoted as G$\otimes$R and is the matrix whose ij-block is given by g_{ij}R. The vector g=vec G=$[g_1' \; g_2' \; ...g_n']'$ is formed by stacking all the columns of the matrix G=$[g_1 g_2 \; ...g_n]$. As is evident from the above, function arguments are often omitted for brevity when no confusion is possible. The notation $G(s) \leq \mathcal{O}(s^\nu)$ means that no entry in $G(s)$ grows faster than s^ν as $s \rightarrow \infty$. In the partial fraction expansion of $G(s)$, the contributions made by all its finite poles in Re s $\leq$ 0, Re s>0, and by its poles at $s = \infty$ are denoted by $\{G\}_+$, $\{G\}_-$, and $\{G\}_\infty$, respectively. Clearly, $\{G\}_+$ is analytic in Re s>0, $\{G\}_-$ in Re s ≤ 0 and both are $\leq \mathcal{O}(s^{-1})$. The identity matrix is denoted by I. A positive definite (positive semidefinite) matrix G is denoted by $G > 0$ ($G \geq 0$). A matrix G is said to be *good* when it is analytic on J, the finite part of the $s = j\omega$-axis. The normal rank of a matrix $G(s)$, denoted $\gamma(G)$, is the highest order of all non-identically-zero minors of $G(s)$. When needed for clarity vertical and/or horizontal lines are used to identify the partitions of a matrix. The expectation operator is denoted by $\mathcal{E}\{\}$.

2. Summary of Available Results

The Wiener-Hopf design of optimal 3DOF systems and the parameterization of all suboptimal designs which is presented in [1] is summarized in this section. In this regard, it is convenient to introduce the three closed-loop transfer matrices

$$R_u = (I + C_w F_t P)^{-1} C_u, \tag{2.1}$$
$$R_w = (I + C_w F_t P)^{-1} C_w, \tag{2.2}$$

and

$$R_z = (I + C_w F_t P)^{-1} C_z. \tag{2.3}$$

Also essential in this work are several coprime polynomial matrix fraction descriptions. Specifically, one can always write [41]

$$F_t P = B_1 A_1^{-1} = A^{-1} B, \tag{2.4}$$

where B_1, A_1 is a right and A, B is a left coprime pair of polynomial matrices for which there exist real polynomial matrices X, Y, X_1, and Y_1 so that

$$\begin{aligned} A_1 Y_1 &= YA, & \text{(2.5a)} \\ AX + BY &= I, & \text{(2.5b)} \\ X_1 A_1 + Y_1 B_1 &= I, & \text{(2.5c)} \end{aligned}$$

and

$$det X det X_1 \not\equiv 0. \tag{2.5d}$$

It is a consequence of Theorem 1 in [1] that any C_w associated with a stabilizing controller is of the form

$$C_w = (X_1 - K_1 B)^{-1}(Y_1 + K_1 A) \tag{2.6}$$

where K_1 is a real rational matrix analytic in Re s$\geq$0. It then easily follows from (2.2) that

$$R_w = A_1(Y_1 + K_1 A) \tag{2.7}$$

and

$$I - R_w F_t P = (I + C_w F_t P)^{-1} = A_1(X_1 - K_1 B). \tag{2.8}$$

The cost functional E which takes tracking error, saturation, and sensitivity into account was introduced in [1,2] and is as follows:

$$\begin{aligned} E &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr(G_e(s))ds + \frac{k}{2\pi j}\int_{-j\infty}^{j\infty} Tr(Q(s)G_r(s))ds \\ &+ \frac{\mu}{2\pi j}\int_{-j\infty}^{j\infty} Tr(\tilde{S}G_s(s)\tilde{S}_*)ds + \frac{\mu}{2\pi j}\int_{-j\infty}^{j\infty} Tr(kQR_wF_tG_sF_{t*}R_{w*})ds \qquad (2.9) \end{aligned}$$

where G_e, G_r, and G_s are the spectral densities of the error $e(s) = T_0(s)u(s) - y(s)$, the input $r(s)$, and the plant uncertainty (see equation (2.18)), respectively. The constants k and μ are nonnegative and $Q(s)$ is a para-Hermitian nonnegative definite weighting matrix. The matrix $\tilde{S}(s) = I - PR_wF_t$ is a closed-loop sensitivity matrix.

The expression for E in (2.9) decomposes into

$$E = E_u + E_{wz} \tag{2.10}$$

in which

$$\begin{aligned} E_u &= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}[Tr\{kQR_uG_uR_{u*} + (P_*P + kQ)R_uG_nR_{u*}\} \\ &\quad + Tr\{(T_0 - PR_u)G_u(T_0 - PR_u)_*\}]\,ds \end{aligned} \tag{2.11}$$

includes all terms involving R_u and

$$\begin{aligned} E_{wz} = &\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr[kQ(R_wF_tP_0 + R_zL_t)G_d(R_wF_tP_0 + R_zL_t)_* \\ + &\quad (P_*P + kQ)(R_wG_mR_{w*} + R_zG_lR_{z*}) + kQR_wF_t\mu G_sF_{t*}R_{w*}]\,ds \\ + &\quad \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr[(P_0 - PR_wF_tP_0 - PR_zL_t)G_d(P_0 - PR_wF_tP_0 - PR_zL_t)_* \\ + &\quad (I - PR_wF_t)\mu G_s(I - PR_wF_t)_*]\,ds \end{aligned} \tag{2.12}$$

all those involving R_w and R_z. In (2.11) and (2.12), G_d, G_l, G_m, G_u, and G_n are specified spectral densities for the signals $d(s)$, $l(s)$, $m(s)$, $u(s)$, and $n(s)$ respectively, which also account for modelling of any shape-deterministic components in the signals [4].

Clearly, E_u is a nonnegative functional of R_u and E_{wz} is a nonnegative functional of R_w and R_z. So the minimization of E is equivalent to two separate minimization problems: the minimization of E_u with respect to R_u and the minimization of E_{wz} with respect to the pair R_w, R_z. Also of interest is the parameterization of the set of all R_u, R_w, and R_z associated with an asymptotically stable 3DOF system possessing a finite E_u and a finite E_{wz}. In this regard, the two definitions which follow are introduced.

Definition 1: The real rational matrices R_u, R_w, and R_z are said to be *acceptable* for the given F_t, P if there exists a controller which realizes them as the designated closed-system transfer matrices of an internally asymptotically stable configuration of the generic type shown in Fig. 1.1.

Definition 2: The acceptable matrices R_u, R_w, and R_z are called *admissible* if they yield finite cost E. An admissible controller is one which realizes an admissible triple R_u, R_w, and R_z.

The admissible R_u for which E_u is minimized is designated by $\tilde{R}_u$. Similarly, the admissible pair R_w, R_z for which E_{wz} is minimized is designated by $\tilde{R}_w$, $\tilde{R}_z$. Specific formulas for $\tilde{R}_u$, $\tilde{R}_w$, and $\tilde{R}_z$ are given in [1]. The associated minimum values for E_u and E_{wz} are accordingly indicated by $\tilde{E}_u$ and $\tilde{E}_{wz}$, respectively.

It is evident from (2.2) that C_w is determined from R_w by the formula

$$C_w = (I - R_w F_t P)^{-1} R_w. \tag{2.13}$$

So there is a one-to-one correspondence between C_w and R_w. Clearly, only C_w impacts on the stability margin of the feedback loop. Hence, no further consideration is given here to E_u and R_u. Interested readers are referred to [42,43] where suboptimal choices of R_u are investigated in connection with the design of decoupled multivariable systems.

Before one can describe in detail the results obtained in [1] that are needed here, two Wiener-Hopf spectral factorizations have to be defined. In the first, the matrix Λ is a Wiener-Hopf spectral solution (i.e., Λ and Λ^{-1} are analytic in Re s>0) of the equation

$$A_{1*}(P_* P + kQ)A_1 = \Lambda_* \Lambda. \tag{2.14}$$

The second Wiener-Hopf spectral factorization involves the matrices

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \tag{2.15}$$

and

$$\Phi = \begin{bmatrix} F_t G_{ds} F_{t*} + G_m & F_t P_0 G_d L_{t*} \\ L_t G_d P_{0*} F_{t*} & L_t G_d L_{t*} + G_l \end{bmatrix} \tag{2.16}$$

where the matrix G_{ds} is given by

$$G_{ds} = P_0 G_d P_{0*} + \mu G_s \tag{2.17}$$

and

$$G_s = \mathcal{E}\{(\delta P)(\delta P)_*\} + \mathcal{E}\{(\delta P_0)(\delta P_0)_*\} \tag{2.18}$$

is a probabilistic model for first-order perturbations δP and δP_0 in the plant transfer matrices P and P_0, respectively. The matrix G_s appears in the terms included in the cost functional E which take into account the sensitivity of the tracking error e and plant input r to plant parameter changes. In the second Wiener-Hopf spectral factorization needed, the matrix Ω is a Wiener-Hopf spectral solution of the equation

$$\tilde{A}\Phi\tilde{A}_* = \Omega\Omega_*. \tag{2.19}$$

A set of seven underlying assumptions cited below are introduced in [1] for which the following holds.

(i) The set of all acceptable R_w and R_z that yield a finite E_{wz} is generated by the formula

$$[R_w|R_z] = [\tilde{R}_w|\tilde{R}_z] + A_1\Lambda^{-1}[Z_w|Z_z]\Omega^{-1}\tilde{A} \tag{2.20}$$

where Z_w and Z_z can be any real rational matrices analytic in Re s$\geq$0 and satisfying $Z_w \leq \mathcal{O}(s^{-1})$, $Z_z \leq \mathcal{O}(s^{-1})$.

(ii) The cost functional E_{wz} is given by

$$E_{wz} = \tilde{E}_{wz} + \Delta E_{wz} \tag{2.21}$$

where

$$\Delta E_{wz} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr[Z_w Z_{w*} + Z_z Z_{z*}]ds. \tag{2.22}$$

Clearly, the choice $Z_w \equiv 0$, $Z_z \equiv 0$ leads to optimum performance. When optimum performance exceeds specifications, there exists a number N for which all allowed

$$Z_{wz} = [Z_w|Z_z] \tag{2.23}$$

satisfying

$$\Delta E_{wz} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr[Z_{wz} Z_{wz*}]ds \ \leq \ N \tag{2.24}$$

can be considered for design trade offs. Here, in particular, that allowed Z_{wz} satisfying equation (2.24) is sought for which a stability margin measure is minimized. The purpose of this paper is to describe a mechanism for achieving this goal.

The set of seven underlying assumptions mentioned above also insure the existence of an admissible R_u so that a finite E is guaranteed to exist. This assures that the problem addressed here is a meaningful one. These seven assumptions include conditions which also involve the specified spectral densities G_u and G_n for the signals $u(s)$ and $n(s)$, respectively. For easy reference, these seven assumptions follow.

Assumption 1: All system blocks are free of hidden poles in Re s≥0. The transfer matrix L_t(s) and any portion of P_0(s) arising from a block outside the feedback loop are analytic in Re s≥0. For any finite pole s_0 of P or F_t in Re s≥0, the McMillan degree of s_0 as a pole of F_tP is equal to the sum of its McMillan degrees as a pole of P and F_t.

Assumption 2: The matrix $T_0(s)$ is proper and the spectral density $G_u(s)$ vanishes at least as fast as $1/s^2$, as $s \to \infty$, i.e., $G_u(s) \leq \mathcal{O}(s^{-2})$.

Assumption 3: T_0 and F_t are analytic and nonsingular on the finite part of the $s = j\omega$ axis. In addition, T_0 is analytic in Re s>0 and $(T_0F_t - I)P$ is analytic on the finite part of the $s = j\omega$ axis.

Assumption 4: AG_uA_*, Q, G_n, and $(G_u + G_n)^{-1}$ are good (i.e., analytic on the finite part of the $s = j\omega$ axis).

Assumption 5: $A_{1*}(P_*P + kQ)A_1$ is nonsingular on the finite part of the $s = j\omega$ axis.

Assumption 6: The matrix $\tilde{A}\Phi\tilde{A}_*$ is analytic and nonsingular on the finite part of the $s = j\omega$ axis.

Assumption 7: The plant-uncertainty spectral density $G_s(s)$ and $P_0(s)G_d(s)P_{0*}(s)$ are both $\leq \mathcal{O}(s^{-2})$, and $G_m(s)$ is analytic on the finite part of the $s = j\omega$ axis. In addition $F_tP \leq \mathcal{O}(s^{-\nu_0})$, $\Phi^{-1}(s) \leq \mathcal{O}(s^{-2\nu_1})$, and $(P_*P + kQ)^{-1} \leq \mathcal{O}(s^{-2\nu_2})$, where $\nu_0 + \nu_1 + \nu_2 \geq 0$.

An additional assumption which plays an essential role in the sequel is now introduced.

Assumption 8: $\nu_1 + \nu_2 + 1 \geq 0$.

It is now possible to state an important lemma which is an immediate consequence of Lemma 2 in [1].

Lemma 1: When Assumptions 7 and 8 hold, then $\Omega^{-1}\tilde{A} \leq \mathcal{O}(s^{-\nu_1})$, $A_1\Lambda^{-1} \leq \mathcal{O}(s^{-\nu_2})$, and all admissible R_w, R_z satisfy $[R_w \mid R_z] \leq \mathcal{O}(s^{-(\nu_1+\nu_2+1)})$ which implies that R_w, R_z are proper.

It is also convenient here to introduce the following definition.

Definition 3: Any pair of transfer matrices F_t, P that satisfy Assumption 1 are called admissible.

3. Stability Margin Considerations and Problem Formulation

The transfer matrices shown in Fig. 1.1 are nominal transfer matrices. Except for the plant, it is assumed that the nominal and actual transfer matrices of the system components are the same. The plant, however, has an actual transfer matrix which differs from the nominal one. In this paper, there are two differences between the nominal plant transfer matrix and the actual plant transfer matrix that are of interest. The difference δP in (2.18) represents the small uncertainty of the plant when it is operating normally near nominal conditions. The difference $\hat{\delta P}$ is introduced in the sequel to characterize large perturbations associated with abnormal operation that could lead to instability. The overall objective here is to design the controller so that with normal system performance a specification on the value of the cost functional E_{wz} is met, while providing the maximum tolerance against instability arising from the perturbations $\hat{\delta P}$. In this section, attention is focused on the question of stability and the impact of $\hat{\delta P}$ on this issue. In this regard, the right coprime polynomial matrix fraction description

$$F_t(P + \hat{\delta P}) = (B_1 + \delta B_1)(A_1 + \delta A_1)^{-1} \tag{3.1}$$

is introduced to define the polynomial perturbations δA_1 and δB_1.

When Assumption 1 holds and the pair F_t, $(P+\hat{\delta P})$ is admissible, it is not difficult to show [1] that the system is internally asymptotically stable for perturbations δA_1 and δB_1 if, and only if,

$$\phi_\delta = det[I + (X_1 - K_1 B)\delta A_1 + (Y_1 + K_1 A)\delta B_1] \tag{3.2}$$

is free of zeros in Re s$\geq$0. Using (2.7) and (2.8), (3.2) may immediately be written as

$$\phi_\delta = det\{I + A_1^{-1} \begin{bmatrix} \hat{S} & R_w \end{bmatrix} \begin{bmatrix} \delta A_1 \\ \delta B_1 \end{bmatrix}\} \tag{3.3}$$

where

$$\hat{S} = I - R_w F_t P \tag{3.4}$$

is a loop sensitivity matrix on account of (2.8).

Since all coprime polynomial matrix pairs associated with matrix fraction descriptions of the same rational matrix differ from one another only by a unimodular matrix multiplier [44,45], the results cited above are independent of the particular choice of matrix fraction descriptions used (e.g., see the final paragraph in the Appendix of [2]). However, it is essential here that A_1 satisfies the following assumption.

Assumption 9: The polynomial matrix A_1 is column reduced and A_1^{-1} is strictly proper.

Assumption 9 is not overly restrictive. In fact, given any coprime matrix fraction description $\hat{B}\hat{A}^{-1}$, one can find a unimodular matrix U so that

$$\hat{B}_1\hat{A}_1^{-1} = (\hat{B}_1 U)(\hat{A}_1 U)^{-1} = B_1 A_1^{-1} \tag{3.5}$$

where A_1 is column reduced [44,45]. So there is no loss in generality in assuming that A_1 is column reduced. It is also true in almost all cases of practical interest that A_1^{-1} is strictly proper when A_1 is column reduced. Indeed, one has the following lemma.

Lemma 2: When $F_t P = B_1 A_1^{-1}$ is strictly proper, when the pair B_1, A_1 is right coprime with A_1 column reduced, and when there exists no constant vector $\alpha \neq 0$ for which $F_t P\alpha \equiv 0$, then A_1^{-1} is strictly proper.

Proof: First, it is established that the column degrees of A_1 must all be positive. When the contrary is true, there is at least one column of A_1 which has zero degree. This column then contains only constants. These constants cannot all be zero because det $A_1 \not\equiv 0$. This constant nonzero vector is designated by α. When $(B_1)_j$ denotes the j-column of B_1 and α is the j-column of A_1, it then follows that

$$(B_1)_j = F_t P\alpha. \tag{3.6}$$

Since F_tP is strictly proper,

$$\lim_{s \to \infty} (B_1)_j = 0. \tag{3.7}$$

But $(B_1)_j$ is polynomial and the only way (3.7) can be true is if $(\mathrm{B}_1)_j \equiv 0$. It then follows from (3.6) that

$$F_tP\alpha \equiv 0, \ \alpha \neq 0 \tag{3.8}$$

which contradicts the hypothesis. It is now a simple matter to prove A_1^{-1} is strictly proper. Since A_1 is column reduced, one can write

$$A_1 = (A_1)_{hc}S + L \tag{3.9}$$

where $(A_1)_{hc}$ is a nonsingular constant matrix,

$$S = diag\{s^{\delta_{c1}}, s^{\delta_{c2}}, ..., s^{\delta_{cm}}\} \tag{3.10}$$

with δ_{cj} being the column degree of the j-column of A_1, and the column degrees of the polynomial matrix L are less than the corresponding column degrees of A_1. Clearly,

$$\lim_{s \to \infty} A_1^{-1} = \lim_{s \to \infty} S^{-1}[(A_1)_{hc} + LS^{-1}]^{-1} = \lim_{s \to \infty} S^{-1}(A_1)_{hc}^{-1} = 0 \tag{3.11}$$

and so A_1^{-1} is strictly proper. This completes the proof.

It should be noted that lemmas similar to Lemma 2 have appeared in the literature (e.g., see Theorem G-10 in [44] and Lemma 6.3-11 in [45]), but the possibility of (3.8) was overlooked. A simple example which exposes the difficulty is easily generated from

$$\begin{bmatrix} B_1 \\ A_1 \end{bmatrix} = \left[\begin{array}{cc} 0 & 1 \\ \hline 1 & s \\ 1 & 2s \end{array}\right]. \tag{3.12}$$

In this case, $F_tP = \begin{bmatrix} \frac{-1}{s} & \frac{1}{s} \end{bmatrix}$ is strictly proper; however, A_1^{-1} is only proper even though A_1 is column reduced. This example is ruled out as a counterexample in Lemma 2 because it satisfies $F_tP\alpha \equiv 0$ for $\alpha = [1 \quad 1]'$.

It should be emphasized that Lemma 2 provides sufficient conditions for A_1^{-1} to be strictly proper. In fact, one frequently finds that A_1^{-1} is strictly proper when F_tP is only proper or even improper.

The modelling of the perturbations δA_1, δB_1 is considered next. Since A_1 is column reduced, one can always write

$$A_1 = \sum_{i=0}^{k} A_{1i} S_i, \quad det A_{1k} = det(A_1)_{hc} \neq 0 \tag{3.13}$$

where the elements of the A_{1i} are real numbers,

$$k = max\{\delta_{c1}, \delta_{c2}, ..., \delta_{cm}\} \tag{3.14}$$

and

$$S_i = diag\{s^{\delta_{i_1}}, s^{\delta_{i_2}}, ..., s^{\delta_{i_m}}\} \tag{3.15}$$

with

$$\delta_{i_j} = max\{\delta_{c_j} - k + i, 0\}. \tag{3.16}$$

It should be noted that $S_k = S$. Attention is restricted here to perturbations of the form

$$\begin{bmatrix} \delta A_1 \\ \delta B_1 \end{bmatrix} = (\delta \mathcal{P}) \mathcal{S} \tag{3.17}$$

where

$$\delta \mathcal{P} = \begin{bmatrix} \delta A_{10} & \delta A_{11} & \cdots & \delta A_{1,k-1} \\ \delta B_{10} & \delta B_{11} & \cdots & \delta B_{1,k-1} \end{bmatrix} \tag{3.18}$$

and

$$\mathcal{S} = [S_0' \; S_1' \cdots S_{k-1}']'. \tag{3.19}$$

The implication of this modelling is treated first. Practical justification then follows.

Substituting (3.17) into (3.3) and making use of

$$det(I + G_1 G_2) = det(I + G_2 G_1) \tag{3.20}$$

gives

$$\phi_\delta = det[I + \mathcal{S}M(\delta\mathcal{P})] \tag{3.21}$$

where

$$M = A_1^{-1} \begin{bmatrix} \hat{S} & R_w \end{bmatrix}. \tag{3.22}$$

Now, because of Assumptions 7 and 8 and Lemma 1,

$$R_w F_t P \leq \mathcal{O}(s^{-(\nu_0+\nu_1+\nu_2+1)}) \leq \mathcal{O}(s^{-1}). \tag{3.23}$$

Hence, $\hat{S}$ is proper for all admissible R_w. Also, since $S = S_k$,

$$\lim_{s\to\infty} \mathcal{S}A_1^{-1} = \lim_{s\to\infty}\{\begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_{k-1} \end{bmatrix} S_k^{-1}[(A_1)_{hc} + LS_k^{-1}]^{-1}\} = 0. \tag{3.24}$$

So $\mathcal{S}M$ in (3.21) is strictly proper for all admissible R_w. Moreover, the matrix M is analytic in Re s$\geq$0 since

$$A_1^{-1}[\hat{S} \mid R_w] = [(X_1 - K_1B) \mid (Y_1 + K_1A)]. \tag{3.25}$$

Hence, $\mathcal{S}M(\delta\mathcal{P})$ is analytic in Re s$\geq$0 and strictly proper.

It immediately follows from the generalized Nyquist theorem (e.g., see Chapter 2 of [46]) that ϕ_δ is free of zeros in Re s$\geq$0 when there are no encirclements in the complex s-plane of the point $-1 + j0$ by the characteristic loci for $\mathcal{S}(j\omega)M(j\omega)(\delta\mathcal{P})$. Now M is a function of R_w which in turn is a function of Z_{wz}. So one should seek an allowed Z_{wz} satisfying (2.24) for which there are no encirclements of the $-1+j0$ point by the characteristic loci for the largest set of expected perturbations $\delta\mathcal{P}$ (i.e., so that the stability margin is maximized). This design problem defies an analytical solution, however. Hence, an alternative design approach is considered here. Specifically, that allowed Z_{wz} satisfying (2.24) is sought for which

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr(\mathcal{S}M\Sigma M_*\mathcal{S}_*)ds \tag{3.26}$$

is minimized where with $\mathcal{E}$ denoting the expectation operator

$$\Sigma = \mathcal{E}\{(\delta\mathcal{P})(\delta\mathcal{P})'\}. \tag{3.27}$$

Clearly, in this formulation the perturbation matrix is assumed to contain elements which are zero-mean random variables with known covariances. When this is not the case, one might take $\Sigma = I$ or one might replace Σ with some appropriate weighting matrix. The advantage of including Σ in J is that when Σ is known, the structure of the perturbations is in some way included in the design process. J being finite for all allowed Z_{wz} is a consequence of the fact that $\mathcal{S}M$ is analytic in Re s$\geq$0 and strictly proper for all admissible R_w.

It is expected that this alternative approach may yield designs with stability margins that are significantly better than those for the optimal performance design which corresponds to the choice $Z_{wz} \equiv 0$. This expectation is based on the anticipation that designs for which J is kept small are designs which in practice tend to keep $\mathcal{S}(j\omega)M(j\omega)\delta\mathcal{P}$ small so that encirclement of the $-1+j0$ point are avoided.

The functional J is a special case of the functional

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr(W_1 M W_2 M_* W_{1*})ds \tag{3.28}$$

where W_1 and W_2 satisfy the following assumption.

Assumption 10: The $km \times m$ matrix W_1 is analytic in $0 \leq$Re s$< \infty$ and such that $W_1 \mathrm{A}_1^{-1}$ is strictly proper. The (n+m)$\times$(n+m) matrix W_2 is a proper para-Hermitian matrix which is analytic and nonnegative definite on the $s = j\omega$ axis.

In order that the theory developed here be applicable to situations in which the modelling of the perturbations δA_1, δB_1 is done differently and/or the weighting matrix Σ in (3.26) is chosen differently, all results in the sequel are developed for the more general case of (3.28). Specifically, a formula is derived which gives the strictly proper Z_{wz} analytic in Re s$\geq$0 and satisfying (2.24) for which (3.28) is minimized.

The practicality of the perturbation model assumed in (3.17) thru (3.19) is justified next. In most applications, $F_t P$ is strictly proper. When A_1 is column reduced, it then follows that

$$\lim_{s\to\infty} F_t P = \lim_{s\to\infty} B_1 S_k^{-1} A_{1k}^{-1} = 0. \tag{3.29}$$

Now (3.29) holds if, and only if,

$$\lim_{s\to\infty} B_1 S_k^{-1} = 0 \tag{3.30}$$

or the column degrees of B_1 are less than the corresponding column degrees of A_1.

In this case, it is always possible to write

$$B_1 = \Sigma_{i=o}^{k-1} B_{1i} S_i \tag{3.31}$$

and

$$\delta B_1 = \Sigma_{i=o}^{k-1} \delta B_{1i} S_i \tag{3.32}$$

which agrees with the second block row of (3.17).

The justification for taking $\delta A_{1k} = 0$ is more involved. When $\delta A_{1k} \neq 0$, then

$$\delta \mathcal{P} = \begin{bmatrix} \delta \mathcal{P}_A & \delta A_{1k} \\ \delta \mathcal{P}_B & 0 \end{bmatrix} \tag{3.33}$$

where

$$\begin{bmatrix} \delta \mathcal{P}_A \\ \delta \mathcal{P}_B \end{bmatrix} = \begin{bmatrix} \delta A_{10} & \delta A_{11} & \cdots & \delta A_{1,k-1} \\ \delta B_{10} & \delta B_{11} & \cdots & \delta B_{1,k-1} \end{bmatrix} \tag{3.34}$$

and

$$\begin{bmatrix} \delta A_1 \\ \delta B_1 \end{bmatrix} = \begin{bmatrix} \delta \mathcal{P}_A & \delta A_{1k} \\ \delta \mathcal{P}_B & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S} \\ S_k \end{bmatrix}. \tag{3.35}$$

Using (3.35) in (3.3) and invoking (3.20) now yields

$$\phi_\delta = det(I + \mathcal{M}) \tag{3.36}$$

where

$$\mathcal{M} = \begin{bmatrix} \mathcal{S} \\ S_k \end{bmatrix} A_1^{-1} \begin{bmatrix} \hat{S} & R_w \end{bmatrix} \begin{bmatrix} \delta \mathcal{P}_A & \delta A_{1k} \\ \delta \mathcal{P}_B & 0 \end{bmatrix}. \tag{3.37}$$

Since under Assumptions 7 and 8, R_w and $F_t P R_w$ are strictly proper for all admissible R_w,

$$\lim_{s \to \infty} \begin{bmatrix} \hat{S} & R_w \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{3.38}$$

It should now be clear when (3.24) is recalled that

$$\lim_{s \to \infty} \mathcal{M} = \begin{bmatrix} 0 \\ A_{1k}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \delta \mathcal{P}_A & \delta A_{1k} \\ \delta \mathcal{P}_B & 0 \end{bmatrix} \tag{3.39}$$

or

$$\mathcal{M}(\infty) = \begin{bmatrix} 0 & 0 \\ A_{1k}^{-1}\delta\mathcal{P}_A & A_{1k}^{-1}\delta A_{1k} \end{bmatrix}. \tag{3.40}$$

So with $\delta A_{1k} \neq 0$, $\mathcal{M}$ is not strictly proper and any H_2 norm associated with this matrix is infinite.

Obviously, $\delta A_{1k} = 0$ is needed for the theory developed here. On the other hand, since δA_{1k} impacts primarily on the high frequency behavior of $\mathcal{M}$ and since from (3.40) the eigenvalues of $\mathcal{M}(\infty)$ are all less than unity in magnitude when

$$\|A_{1k}^{-1}\delta A_{1k}\| \leq \|A_{1k}^{-1}\|\|\delta A_{1k}\| < 1, \tag{3.41}$$

it is clear that the impact of non-null δA_{1k} on encirclements of the $-1 + j0$ point by the characteristic loci of $\mathcal{M}$ is small when the inequality in (3.41) is valid. So the designs proposed here are more robust to parameter variations that affect the low frequency behavior of the plant. But this is usually desired, since loop gains are typically small in the higher frequency range. The problem formulation is now complete.

4. The Analytical Solution

The constrained optimization problem associated with (2.24) and (3.28) is solved by introducing a Lagrangian [16,17] multiplier α^2 and minimizing first

$$\hat{J} = J + \alpha^2 \Delta E_{wz}. \tag{4.1}$$

This yields the minimizing Z_{wz} as a function of α^2. The parameter α^2 is then chosen so that (2.24) is satisfied. It is shown in the Appendix A that J and ΔE_{wz} are a monotonically increasing and a monotonically decreasing function of α^2, respectively. Also, it should be clear that minimizing $\hat{J}$ is equivalent to minimizing

$$\lambda_1 \hat{J} = \lambda_1 J + \lambda_2 \Delta E_{wz} \tag{4.2}$$

when

$$\lambda_1 = \frac{1}{1+\alpha^2} \quad , \quad \lambda_2 = \frac{\alpha^2}{1+\alpha^2} = 1 - \lambda_1. \tag{4.3}$$

So the set of all Z_{wz} that minimize $\hat{J}$ for $0 \leq \alpha^2 < \infty$ is the same set that minimizes $\lambda_1 \hat{J}$ with $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$. Hence, as described in [31,32], the set of solutions Z_{wz} is the set of Pareto optimal solutions for the multiple objective criteria J and ΔE_{wz}. In fact, after some aditional observations described below, it is possible to easily derive a formula for this set of Z_{wz} by using equation (4.5) in [31] and setting $\alpha^2 = \frac{\lambda_2}{\lambda_1}$ in the result. This is done in Appendix B.

Because of the properties of W_2 included in Assumption 10, one can always write

$$W_2 = H_2 H_{2*} \tag{4.4}$$

where H_2 is proper and analytic in $Re(s) \geq 0$. It then follows from (3.28) that

$$J = ||T_1||_2^2 \triangleq \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} Tr(T_1 T_{1*}) ds \tag{4.5}$$

where

$$T_1 = W_1 M H_2. \tag{4.6}$$

It is also obvious from (2.24) that

$$\Delta E_{wz} = ||T_2||_2^2 \tag{4.7}$$

where

$$T_2 = Z_{wz}. \tag{4.8}$$

Moreover, the matrix M in (4.6) can be expressed in terms of Z_{wz} with the aid of (2.20) and (3.22). From (2.20),

$$R_w = \tilde{R}_w + A_1 \Lambda^{-1} Z_{wz} \Omega^{-1} \begin{bmatrix} A \\ 0 \end{bmatrix}. \tag{4.9}$$

Then, when (2.4) is recalled,

$$\hat{S} = I - R_w F_t P = I - \tilde{R}_w F_t P - A_1 \Lambda^{-1} Z_{wz} \Omega^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{4.10}$$

Thus, (3.22) gives

$$M = A_1^{-1} \left[\begin{array}{cc} \hat{S} & R_w \end{array} \right] = \Psi_1 + \Psi_2 Z_{wz} \Psi_3 \tag{4.11}$$

where

$$\Psi_1 = A_1^{-1}[(I - \tilde{R}_w F_t P) \mid \tilde{R}_w], \tag{4.12}$$

$$\Psi_2 = \Lambda^{-1}, \tag{4.13}$$

and

$$\Psi_3 = \Omega^{-1} \left[\begin{array}{cc} -B & A \\ 0 & 0 \end{array} \right]. \tag{4.14}$$

Hence (4.6) is equivalent to

$$T_1 = R_1 - U_1 Z_{wz} V_1 \tag{4.15}$$

where

$$R_1 = W_1 \Psi_1 H_2, \tag{4.16}$$

$$U_1 = -W_1 \Psi_2, \tag{4.17}$$

and

$$V_1 = \Psi_3 H_2. \tag{4.18}$$

Similarly, (4.8) is of the form

$$T_2 = R_2 - U_2 Z_{wz} V_2, \tag{4.19}$$

with

$$R_2 = 0, \quad U_2 = -I, \quad V_2 = I. \tag{4.20}$$

Clearly, R_2, U_2, and V_2 are proper, real, rational matrices which are analytic in $Re(s) \geq 0$. It is also established in the sequel that the same is true of R_1, U_1, and V_1. In addition, both R_1 and R_2 are strictly proper. As a consequence, minimizing (4.2) or

$$\lambda_1 \hat{J} = ||\sqrt{\lambda_1} R_1 - \sqrt{\lambda_1} U_1 Z_{wz} V_1||_2^2 + ||\sqrt{\lambda_2} R_2 - \sqrt{\lambda_2} U_2 Z_{wz} V_2||_2^2 \tag{4.21}$$

with respect to Z_{wz} is a special case of the problem already solved in Sections III and IV of [31]. Here Z_{wz} plays the role of the matrix Q in [31] and the fact that R_1 and R_2 are strictly proper guarantees that Z_{wz} will also be.

That R_1, U_1, and V_1 have the properties described above is established with the aid of the following lemmas and one additional assumption.

Lemma 3: $W_1 \Psi_2 \leq \mathcal{O}\left(s^{(-\nu_2+1)}\right)$.

Proof: From (4.13),

$$W_1 \Psi_2 = W_1 \Lambda^{-1} = (W_1 A_1^{-1})(A_1 \Lambda^{-1}). \tag{4.22}$$

From Lemma 1, $A_1\Lambda^{-1} \leq \mathcal{O}(s^{-\nu_2})$. Since $W_1 A_1^{-1}$ is strictly proper by Assumption 10, it immediately follows that the lemma is true.

Lemma 4: With $\nu_4 = \max\{0, -\nu_0\}$, $\Psi_3 \leq \mathcal{O}\left(s^{(\nu_4 - \nu_1)}\right)$.

Proof: From (2.4) and (4.14),

$$\Psi_3 = \Omega^{-1} \begin{bmatrix} -AF_tP & A \\ 0 & 0 \end{bmatrix} = \Omega^{-1} \tilde{A} \begin{bmatrix} -F_tP & I \\ 0 & 0 \end{bmatrix}. \tag{4.23}$$

Since

i) $\Omega^{-1}\tilde{A} \leq \mathcal{O}(s^{-\nu_1})$ because of Lemma 1,

ii) $F_tP \leq \mathcal{O}(s^{-\nu_0})$ because of assumption 7,

it immediately follows that the lemma is true.

Lemma 5: Ψ_1, Ψ_2 and Ψ_3 are analytic in $Re(s) \geq 0$.

Proof: Since M is analytic in Re s ≥ 0 because of (3.25) and since it is clear from (4.11) and (4.12) that Ψ_1 is the particular value of M corresponding to the choice $R_w = \tilde{R_w}$, it follows that Ψ_1 is analytic in Re s ≥ 0. The matrix Ψ_3 in (4.14) is analytic in Re s ≥ 0 because Assumption 6 guarantees that Ω^{-1} is good and, therefore, analytic in Re s ≥ 0. Finally, attention is turned to $\Psi_2 = \Lambda^{-1}$. From Assumption 5, $\Lambda_*\Lambda$ is nonsingular on the finite $s = j\omega$ axis. The product $\Lambda_*\Lambda$ is also good because $A_{1*}(P_*P + kQ)A_1 = (PA_1)_*(PA_1) + kA_{1*}QA_1$ good follows from Q good and PA_1 good. That Q is good is a consequence of Assumption 4. The proof that PA_1 is good is given in Section II of [41]. So $\Lambda_*\Lambda$ is good and nonsingular on the finite $s = j\omega$ axis. It immediately follows that the same is true for Λ and $\Psi_2 = \Lambda^{-1}$. Since Λ^{-1} is already analytic in Re s > 0, it follows that Λ^{-1} is analytic in Re s ≥ 0.

The final assumption needed is

Assumption 11: $\nu_o = \nu_1 = \nu_2 = 0$.

Remark: Assumption 11 is consistent with Assumptions 7 and 8 and the specifications in practical cases. Typically, F_tP is proper which is equivalent to $\nu_o = 0$. Also, P is strictly proper and kQ is proper and positive definite at infinity. Hence, $(P_*P + kQ)^{-1}$ is proper which is equivalent to $\nu_2 = 0$. Moreover, the matrix Φ in (2.16) is typically proper and positive definite at infinity so that Φ^{-1} is proper. This is equivalent to $\nu_1 = 0$. Finally, it should be noted that the results derived in the sequel are derived with a somewhat less restrictive Assumption 11 in [47] using standard Wiener-Hopf arguments. Since Assumption 11 is satisfied in most practical applications, however, the simpler derivation based on the already existing results in [31] is given here.

All is now in place to readily confirm the properties of R_1, U_1 and V_1 cited above. By construction H_2 is proper and analytic in Re s ≥ 0. Hence, R_1 in (4.16) is analytic in Re s ≥ 0 and strictly proper when the same is true of $W_1\Psi_1$. Now W_1 is analytic in Re s ≥ 0 by assumption 10 and Ψ_1 is analytic in Re s ≥ 0 by Lemma 5. Hence, R_1 is analytic in Re s ≥ 0. Also,

$$W_1\Psi_1 = W_1A_1^{-1}[(I - \tilde{R}_wF_tP)|\tilde{R}_w]. \tag{4.24}$$

Now $W_1A_1^{-1}$ is strictly proper by Assumption 10. Because of Assumptions 7 and 11 and Lemma 1, it is also true that $\tilde{R}_w$ and $\tilde{R}_wF_tP$ are strictly proper. Hence, it immediately follows from (4.24) that $W_1\Psi_1$ and, therefore, R_1 is strictly proper.

It is obvious that U_1 in (4.17) is proper because of Assumption 11 and Lemma 3. It is also clear that U_1 is analytic in Re s ≥ 0 since W_1 is by Assumption 10 and Ψ_2 is by Lemma 5. Finally, V_1 is proper and analytic in Re s ≥ 0 when Ψ_3 is because of the properties of H_2. Clearly, Ψ_3 is analytic in Re s ≥ 0 by Lemma 5. Also, Ψ_3 is proper since with Assumption 11 one has $\nu_4 = \nu_1 = 0$ in Lemma 4.

It is now a straightforward process to prove the following theorem which represents the key result of this section.

Theorem: When $\alpha \neq 0$ and Assumptions 1 thru 11 are satisfied, the functional

$$\hat{J} = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [Tr(W_1 M W_2 M_* W_{1*}) + \alpha^2 Tr(Z_{wz} Z_{wz*})] ds \tag{4.25}$$

with

$$M = \Psi_1 + \Psi_2 Z_{wz} \Psi_3 = A_1^{-1}[(I - \tilde{R}_w F_t P) \mid \tilde{R}_w] + \Lambda^{-1} Z_{wz} \Omega^{-1} \begin{bmatrix} -B & A \\ 0 & 0 \end{bmatrix} \tag{4.26}$$

is minimized over the set of all strictly proper Z_{wz} analytic in Re s$\geq$0 if, and only if, Z_{wz} is chosen so that

$$vecZ_{wz} = z_0 = \nabla^{-1}\{\nabla_*^{-1} v\}_+ \tag{4.27}$$

where ∇ is a Wiener-Hopf spectral solution to the equation

$$\mathcal{D} = (\Psi_3 W_2 \Psi_{3*})' \otimes (\Psi_{2*} W_{1*} W_1 \Psi_2) + \alpha^2 I = \nabla_* \nabla, \tag{4.28}$$

and where

$$v = vecV = -vec(\Psi_{2*} W_{1*} W_1 \Psi_1 W_2 \Psi_{3*}). \tag{4.29}$$

Proof: See Appendix B.

5. The 2DOF System

It should be clear from Fig.1 that when there is no feedforward transducer (i.e., L_t=0) one should expect that the results obtained for the 3DOF system reduced to a 2DOF system design. That is, one should expect that C_z=0 is obtained. It is demonstrated in this section that this is indeed the case. In the process, the formulas for the 2DOF system design are derived.

It follows from (2.3) that C_z=0 is equivalent to $R_z = 0$. It has already been pointed out in [1] that $\tilde{R}_z = 0$ when $L_t = 0$. Hence, in light of (2.20), it only remains to show that

$$\delta R_z = A_1 \Lambda^{-1} [Z_w | Z_z] \Omega^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = 0. \tag{5.1}$$

The first step entails the derivation of a useful formula for Ω. With L_t=0, one easily finds from (2.15), (2.16), and (2.19) that

$$\Omega \Omega_* = \begin{bmatrix} A(F_t G_{ds} F_{t*} + G_m) A_* & 0 \\ 0 & G_l \end{bmatrix}. \tag{5.2}$$

Hence,

$$\Omega = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \tag{5.3}$$

where Ω_1 and Ω_2 are Wiener-Hopf spectral solutions to

$$\Omega_1 \Omega_{1*} = A(F_t G_{ds} F_{t*} + G_m) A_* \tag{5.4}$$

and

$$\Omega_2 \Omega_{2*} = G_l. \tag{5.5}$$

Substituting (5.3) into (5.1) then yields

$$\delta R_z = A_1 \Lambda^{-1} Z_z \Omega_2^{-1}. \tag{5.6}$$

The next step entails the derivation of useful formulas for Z_w and Z_z starting with (4.27). From (4.14) and (5.3),

$$\Psi_3 = \left[\begin{array}{cc} \hat{\Psi}_3' & 0' \end{array} \right]' \tag{5.7}$$

where

$$\hat{\Psi}_3 = \Omega_1^{-1} \left[\begin{array}{cc} -B & A \end{array} \right]. \tag{5.8}$$

It is now convenient to introduce

$$\mathcal{A} = \Psi_{2*} W_{1*} W_1 \Psi_2$$

and

$$\mathcal{B} = \Psi_3 W_2 \Psi_{3*}.$$

Then (5.7) yields

$$\mathcal{B} = \left[\begin{array}{cc} \hat{\mathcal{B}} & 0 \\ 0 & 0 \end{array} \right] \tag{5.9}$$

where

$$\hat{\mathcal{B}} = \hat{\Psi}_3 W_2 \hat{\Psi}_{3*}, \tag{5.10}$$

and it follows from (4.28) that

$$\mathcal{D} = \left[\begin{array}{cc} (\hat{\mathcal{B}}' \otimes \mathcal{A} + \alpha^2 I) & 0 \\ 0 & \alpha^2 I \end{array} \right] \tag{5.11}$$

and

$$\bigtriangledown = \left[\begin{array}{cc} \hat{\bigtriangledown} & 0 \\ 0 & \alpha I \end{array} \right] \tag{5.12}$$

where $\hat{\bigtriangledown}$ is a Wiener-Hopf spectral solution to

$$\hat{\bigtriangledown}_* \hat{\bigtriangledown} = \hat{\mathcal{B}}' \otimes \mathcal{A} + \alpha^2 I = (\hat{\Psi}_3 W_2 \hat{\Psi}_{3*})' \otimes \Psi_{2*} W_{1*} W_1 \Psi_2 + \alpha^2 I. \tag{5.13}$$

The final piece of information needed for (4.27) is v which follows from (4.29). Clearly, (5.7) leads to

$$V = -\Psi_{2*}W_{1*}W_1\Psi_1W_2\Psi_{3*} = \begin{bmatrix} \hat{V} & 0 \end{bmatrix} \tag{5.14}$$

where

$$\hat{V} = -\Psi_{2*}W_{1*}W_1\Psi_1W_2\hat{\Psi}_{3*}. \tag{5.15}$$

Using careful bookkeeping and recalling that $F_tP = A^{-1}B = B_1\ A_1^{-1}$ is an $n\times$m matrix, one can show that Z_w and $\hat{V}$ are m$\times$n matrices and $\hat{\nabla}$ is an nm$\times$nm matrix. Hence,

$$v = vecV = \begin{bmatrix} \hat{v}' & 0' \end{bmatrix}' \tag{5.16}$$

where the vector

$$\hat{v} = \begin{bmatrix} v_1' & v_2' & \cdots & v_n' \end{bmatrix}' = vec\hat{V} \tag{5.17}$$

has nm rows. Substituting (5.12) and (5.16) into (4.27) then gives

$$vec[Z_w|Z_z] = \begin{bmatrix} \hat{\nabla}^{-1}\{\hat{\nabla}_*^{-1}\hat{v}\}_+ \\ 0 \end{bmatrix} \tag{5.18}$$

from which it follows that

$$vecZ_w = \hat{\nabla}^{-1}\{\hat{\nabla}_*^{-1}\hat{v}\}_+ \tag{5.19}$$

and

$$vecZ_z = 0. \tag{5.20}$$

Hence, (5.6) gives δR_z=0 and (5.1) is confirmed.

The Z_w given by (5.19) is denoted by Z_{w0}. The associated expression for R_w is obtained from (2.20), with Z_z=0 and Ω given by (5.3). One gets

$$R_w = R_{w0} = \tilde{R}_w + A_1\Lambda^{-1}Z_{w0}\Omega_1^{-1}A \tag{5.21}$$

where

$$vecZ_{w0} = \hat{\nabla}^{-1}\{\hat{\nabla}_*^{-1}\hat{v}\}_+. \tag{5.22}$$

Using (5.21) in (2.13) then gives the C_w in the 2DOF controller

$$r(s) = C_u(s)u_0(s) - C_w(s)w(s) \tag{5.23}$$

that results.

Since the functional (4.25) reduces to

$$\hat{J} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}[Tr(W_1 M W_2 M_* W_{1*}) + \alpha^2 Tr(Z_w Z_{w*})]ds \tag{5.24}$$

with

$$M = \Psi_1 + \Psi_2 Z_w \hat{\Psi}_3 \tag{5.25}$$

when Z_z=0, it follows that (5.24) is minimized when R_w=R_{w0} and Z_w=Z_{w0}. This is indeed the result one would seek in the robust design of a 2DOF controller.

It should be noted that the above results are independent of G_l so long as Assumption 6 and $\Phi^{-1} \leq \mathcal{O}(s^{-2\nu_1})$ in Assumption 7 are satisfied. Now (5.2) is the expression for $\tilde{A}\Phi\tilde{A}_*$. Clearly, all that is needed for the satisfaction of Assumption 6 is that $A\hat{\Phi}A_*$, where

$$\hat{\Phi} = F_t G_{ds} F_{t*} + G_m, \tag{5.26}$$

be analytic and nonsingular on the finite part of the $s = j\omega$ axis. For then one can always pick a G_l so that the same is true for $\tilde{A}\Phi\tilde{A}_*$. With regard to

$$\Phi^{-1} = \begin{bmatrix} \hat{\Phi}^{-1} & 0 \\ 0 & G_l^{-1} \end{bmatrix} \leq \mathcal{O}(s^{-2\nu_1}) \tag{5.27}$$

all that is needed is $\hat{\Phi}^{-1} \leq \mathcal{O}(s^{-2\nu_1})$ for one can always pick a G_l satisfying $G_l^{-1} \leq \mathcal{O}(s^{-2\nu_1})$.

As a consequence of the above results and observations it is now possible to state without any further proof the following corollary to the main theorem.

Corollary: When $\alpha \neq 0$ and Assumptions 1 thru 11 are satisfied with $A\hat{\Phi}A_*$ replacing $\tilde{A}\Phi\tilde{A}_*$ in Assumption 6 and with $\hat{\Phi}$ replacing Φ in Assumption 7, the functional

$$\hat{J} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}[Tr(W_1 M W_2 M_* W_{1*}) + \alpha^2 Tr(Z_w Z_{w*})]ds \tag{5.28}$$

with

$$M = \Psi_1 + \Psi_2 Z_w \hat{\Psi}_3 \tag{5.29}$$

is minimized over the set of all strictly proper Z_w analytic in Re s$\geq$ 0 if, and only if, Z_w is chosen so that

$$vecZ_w = \hat{\nabla}^{-1}\{\hat{\nabla}_*^{-1}\hat{v}\}_+ \tag{5.30}$$

where $\hat{\nabla}$ is a Wiener-Hopf spectral solution to the equation

$$\hat{\nabla}_*\hat{\nabla} = (\hat{\Psi}_3 W_2 \hat{\Psi}_{3*})' \otimes (\Psi_{2*} W_{1*} W_1 \Psi_2) + \alpha^2 I \tag{5.31}$$

and where

$$\hat{v} = vec\hat{V} = -vec(\Psi_{2*} W_{1*} W_1 \Psi_1 W_2 \hat{\Psi}_{3*}). \tag{5.32}$$

6. Computational Issues

Once R_w and R_z have been determined, the final step in the design process can be taken. Specifically, the controller transfer matrices C_w and C_z can be computed. From (2.2) and (2.3), one gets

$$C_w = (I - R_w F_t P)^{-1} R_w \tag{6.1}$$

and

$$C_z = (I - R_w F_t P)^{-1} R_z. \tag{6.2}$$

These formulas, however, involve a number of cancellations that must be assured despite roundoff error in the numerical computation. Failure to enforce these theoretically required cancellations can lead to a synthesized controller with poles (zeros) where the plant has zeros (poles) in Re s$\geq$0. In such cases, the system is unstable because of unstable hidden modes.

The best way to proceed is to recognize from (2.20) that R_w and R_z are of the form

$$R_w = A_1 H_w \tag{6.3}$$

and

$$R_z = A_1 H_z \tag{6.4}$$

where H_w and H_z are analytic in Re s≥ 0 since the formulas for $\tilde{R}_w$ and $\tilde{R}_z$ given in [1] are also of this form. So one would compute H_w and H_z separately and then recognize that the substitution of (6.3) and (6.4) into (6.1) and (6.2) yields, respectively,

$$C_w = D_w^{-1} H_w \tag{6.5}$$

and

$$C_z = D_w^{-1} H_z \tag{6.6}$$

where in view of (2.4)

$$D_w = A_1^{-1}(I - R_w F_t P) = (I - H_w B_1) A_1^{-1}. \tag{6.7}$$

It is important to recognize that additional cancellations occur in the computation of D_w. Since R_w is acceptable, it follows from (2.7) and (6.3) that

$$H_w = Y_1 + K_1 A. \tag{6.8}$$

Using (2.5c) and (6.8) in (6.7), one gets

$$\begin{aligned} D_w &= [I - (Y_1 + K_1 A) B_1] A_1^{-1} \\ &= [I - Y_1 B_1 - K_1 A B_1] A_1^{-1} \\ &= [X_1 A_1 - K_1 B A_1] A_1^{-1} = X_1 - K_1 B. \end{aligned} \tag{6.9}$$

Since X_1 and B are polynomial matrices and K_1 is analytic in Re s ≥ 0, it follows that D_w is analytic in Re s ≥ 0. Thus, when using the final result in (6.7) to calculate D_w, one must take care to assure that no poles of A_1^{-1} in Re s ≥ 0 are poles of any element of D_w. This is accomplished for each element of D_w by dividing the factor containing the poles of A_1^{-1} in Re s ≥ 0 into the numerator polynomial and ignoring any remainder polynomial as being due to roundoff error.

7. A Numerical Example

A 2DOF design is carried out in this section to illustrate the advocated design methodology for trade-off between performance and stability margin. The following single-input-single-output plant is considered for a pure servo problem

$$P(s) = \frac{s-1}{s(s-2)}. \tag{7.1}$$

The spectral densities of the signals are $G_u = -\frac{1}{s^2}$ and $G_d = G_m = G_n = 0$, i.e., no disturbance, input noise, or measurement noise. The first-order plant perturbations are accounted for by the plant uncertainty spectral density

$$G_s(s) = \frac{\sigma^2(1-s^2)}{s^2(s^2-4)} \tag{7.2}$$

and the constants are taken as $k = Q = F_t = T_0 = \mu = \sigma = 1$. Therefore, $G_{ds} = G_s$.

The optimal 2DOF design has been carried out in [2] and $\tilde{R}_w$ is given by

$$\tilde{R}_w = \frac{s(s-2)[(8+3\sqrt{7})s-1]}{(s+1)(s^2+\sqrt{7}s+1)}. \tag{7.3}$$

This $\tilde{R}_w$ is realized with

$$\tilde{C}_w = \frac{(8+3\sqrt{7})s-1}{s-(5+2\sqrt{7})}. \tag{7.4}$$

For this design,

$$\tilde{E}_w = 68.395 \tag{7.5}$$

which is the minimum value of the cost functional

$$E_w = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr(R_w G_s R_{w*})\, ds + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Tr[(I-PR_w)G_s(I-PR_w)_*]\, ds. \tag{7.6}$$

Equation (7.6) follows from (2.12) with the given data for this example. It should be noted that this E_w reflects only first-order plant uncertainty so that the choice of a non-zero Z_w for improved stability margin corresponds to an increased sensitivity to "small" plant uncertainty as measured by the cost increment ΔE_w. Moreover, with $Z_u \equiv 0$ no change in tracking and plant saturation cost occurs.

The case considered is one in which the normal operating profile involves only small perturbations in the numerator polynomial coefficients as modelled by (7.2). However, an infrequent abnormal operating profile, which results in a significant change in the plant poles at s=0 and s=2 and the plant zero at s=1, is expected. In order to provide more stability margin for the abnormal case, a trade-off of optimal performance under normal operating conditions for greater stability margin in the abnormal case is considered here through a nonzero choice of Z_w. It is assumed that the plant zero and poles change in a fashion such that only δA_{10} and δB_{10} are nonzero. Then, with δP equal to the first block column of (3.18), it follows that (3.19) reduces to $\mathcal{S} = S_0 = 1$ in (3.17). Moreover, with δA_{10} and δB_{10} independent zero-mean random variables each with unity variance, the matrix Σ in (3.27) is an identity matrix. So in (4.25), $W_1 = \mathcal{S} = 1$ and $W_2 = \Sigma = I$. It now follows in a straightforward fashion from (5.31) since

$$\hat{\Psi}_3 \hat{\Psi}_{3*} = \frac{s^4 - 5s^2 + 1}{\sigma^2(1 - s^2)} \tag{7.7}$$

and

$$\Psi_{2*}\Psi_2 = \frac{1}{s^4 - 5s^2 + 1}, \tag{7.8}$$

that

$$\hat{\nabla} = \frac{\alpha\sigma s + \sqrt{1 + \alpha^2\sigma^2}}{\sigma(s+1)}. \tag{7.9}$$

Furthermore,

$$\hat{v} = \frac{(8 + 3\sqrt{7})s - (5 + 2\sqrt{7})}{\sigma(s^2 - 1)(s^2 - \sqrt{7}s + 1)} = -vec(\Psi_{2*}\Psi_1\hat{\Psi}_{3*}). \tag{7.10}$$

It then follows from (5.30) that

$$Z_w = \frac{\sigma\zeta}{\alpha\sigma s + \sqrt{1 + \alpha^2\sigma^2}} \tag{7.11}$$

where

$$\zeta = \frac{13 + 5\sqrt{7}}{(2 + \sqrt{7})(\alpha\sigma + \sqrt{1 + \alpha^2\sigma^2})}. \tag{7.12}$$

The trade-off in performance and stability margin obtained through different choices of α is summarized in the following table where the values listed for the square root of

$$\|M\|_2^2 = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} Tr(MM_*)\, ds = J \tag{7.13}$$

and for

$$\|M\|_\infty = \sup_\omega \bar{\sigma}(M(j\omega)), \tag{7.14}$$

where $\bar{\sigma}(.)$ denotes the maximum singular value of (.), are the ones obtained with the listed C_w that was obtained using the design methodology described in this paper. The last two columns are explained later. Clearly, by giving up performance, a reasonable gain in the stability margin may be achieved. The first row in the table corresponds to the optimal design (7.4). As expected , the smaller α is, the better is the stability margin and the greater is the degradation in performance. The α corresponding to a 100% increase in cost, namely α^*, is 0.165393. In this case, 5.24 db improvement in stability margin is achieved as measured by $\|M\|_\infty$.

As noted earlier, a quadratic measure was chosen as an approximate measure of stability margin in this paper. To show the efficacy of this design methodology, a numerical optimization was carried out to evaluate the best H_∞ design for each row in Table 1. This optimization was carried out over the set of first order Z_w, namely Z_w^1, that give the listed value of ΔE_w since the optimal solution for Z_w in the H_2 setting was also first order. Specifically, the optimization was carried out over Z_w of the form:

$$Z_w = \frac{a_1}{s + a_2}, \qquad a_2 > 0, \tag{7.15}$$

and $a_1 = \pm\sqrt{2a_2\Delta E_w}$ in order to assure that $\|Z_w\|_2^2 = \Delta E_w$.

The last column in Table 1 lists the stability margin for the best H_∞ solution over the set of allowable first order Z_w when a_1 is negative. The column before the last is the same as the last column except that the parameter a_1 is now positive. Obviously, the attained stability margins for the H_2 solutions are *reasonably close* to the H_∞ solutions. In all cases shown for a positive a_1, the H_2 solutions yielded an improvement

α	E_w	ΔE_w (% of $\tilde{E}_w$)	$\|M\|_\infty$ (db)	$\|M\|_2$ (db)	$C_w = g\frac{(s-z)}{(s-p)}$	$\inf_{Z_w^1} \|M\|_\infty$ (db) – $a_1 > 0$	$\inf_{Z_w^1} \|M\|_\infty$ (db) – $a_1 < 0$
∞	68.395	0 (0%)	20.29	14.187	z=0.063, p=10.292 g=15.937	20.29	20.29
1.0	70.328	1.934 (2.83%)	18.786	13.337	z=0.077, p=12.216 g=18.276	17.74	16.66
0.9	71.006	2.611 (3.82%)	18.570	13.213	z=0.080, p=12.590 g=18.731	17.44	16.66
0.8	71.989	3.594 (5.25%)	18.311	13.064	z=0.083, p=13.083 g=19.329	17.07	16.66
0.7	73.451	5.056 (7.39%)	18.003	12.885	z=0.087, p=13.747 g=20.137	16.63	16.66
0.6	75.697	7.302 (10.67%)	17.636	12.670	z=0.091, p=14.675 g=21.265	16.12	16.66
0.5	79.285	10.890 (15.92%)	17.194	12.410	z=0.098, p=16.034 g=22.916	15.49	16.66
0.4	85.352	16.957 (24.79%)	16.671	12.098	z=0.106, p=18.155 g=25.493	14.74	16.66
0.3	96.563	28.168 (41.18%)	16.052	11.723	z=0.116, p=21.813 g=29.939	13.83	16.66
0.2	120.910	52.515 (76.78%)	15.327	11.277	z=0.131, p=29.335 g=39.080	12.72	16.66
α^*	136.790	68.395 (100%)	15.050	11.103	z=0.137, p=34.117 g=44.891	12.28	16.66
0.1	198.237	129.879 (189.9%)	14.490	10.747	z=0.150, p=52.335 g=67.031	11.36	16.66

Table 7.1: Suboptimal designs for improved stability margin for various αs.

in stability margin which was at least 59% of that obtained with the associated H_∞ solution. Moreover, when a better solution than the H_2 solution is desired, then the H_2 solution provides a good initial point for an iterative search for the H_∞ solution. A larger gain in stability margin may be achieved when $\alpha \geq 0.8$ if negative values of a_1 are permitted. However, in this case, the controller needed is second order and the larger stability margin is realized at the expense of an unreasonably fast pole in the controller. Therefore, the resulting controller is not a practical one. Actually, this is another reason why an approximate measure of stability margin may be more suitable since a controller with a very high bandwidth may be needed if one insists on achieving the maximum possible stability margin. In one example, by giving up a small amount of stability margin, one could reduce the controller bandwidth required to a more practical value.

Additional insight regarding the solutions obtained can be gained from an examination of the stability region in the δA_{10}, δB_{10} parameter space. The closed-loop characteristic polynomial for this example with first order controllers of the form

$$C_w(s) = g\frac{s-z}{s-p} \tag{7.16}$$

is given by

$$\Delta_c(s) = (s^2 - 2s + \delta A_{10})(s-p) + g(s-z)(s-1+\delta B_{10}). \tag{7.17}$$

Appealing to the Routh test for the above polynomial gives as the necessary and sufficient conditions for asymptotic stability:

$$\begin{aligned} g - p - 2 &> 0 && (7.18a) \\ (g-2)\,\delta A_{10} + g(g-p+z-2)\,\delta B_{10} + (g-p-2)[2p - g(1+z)] - gz &> 0 && (7.18b) \\ -\,\delta A_{10} - gz\,\delta B_{10} + gz &> 0. && (7.18c) \end{aligned}$$

The inequality (7.18a) is satisfied automatically since all the controllers stabilize the nominal plant. The last two inequalities establish the range of allowable perturbations in the plant parameters and are linear in the unknown parameters δA_{10} and δB_{10}. The set of parameters for which the closed-loop system is asymptotically stable is shown in Figure 7.1 (the hatched area). This set corresponds to the optimal controller (i.e., $\alpha = \infty$). When perturbations along any ray drawn from the origin are equally likely, the shortest distance from the origin to the boundary of the stability region defines the stability margin. This distance is the radius of the circle shown in Figure 7.1. This radius is in agreement with the value of $\|M\|_\infty^{-1}$ obtained from Table 1 for the $\alpha = \infty$ case and is the minimum distance from the origin to the line l_2. The constraint (7.18c) is indeed the limiting factor in this example. As parameter α decreases, the line l_2 rotates counter-clockwise about the point (0,1); therefore, the stability margin increases. The parameter set for $\alpha^* = 0.165393$ is given in Figure 7.2. In this case, the stability margin is larger by a factor of about 2. The circle with the smallest radius corresponds to the optimal design. The second circle corresponds to the suboptimal design and the largest circle corresponds to the best H_∞ design constrained to the set of first order Z_w. It should be pointed out that it is inherently difficult to achieve a good stability margin for this plant [2] since it is unstable and non-minimum phase and the relative distance between the right-half-plane zero and pole is small. Never-the-less, a substantial relative improvement is possible when a trade-off of performance can be considered.

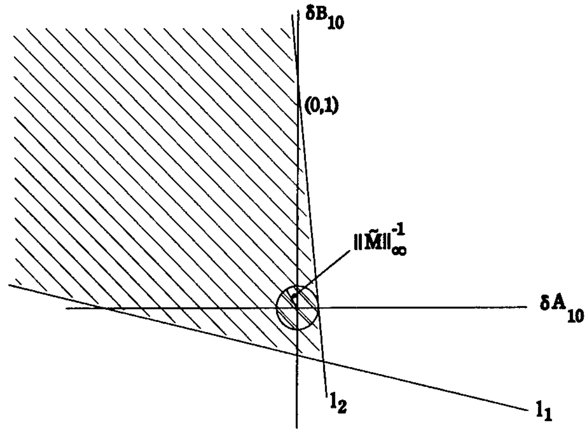

Figure 7.1: Allowable parameter space for the optimal H_2 design.

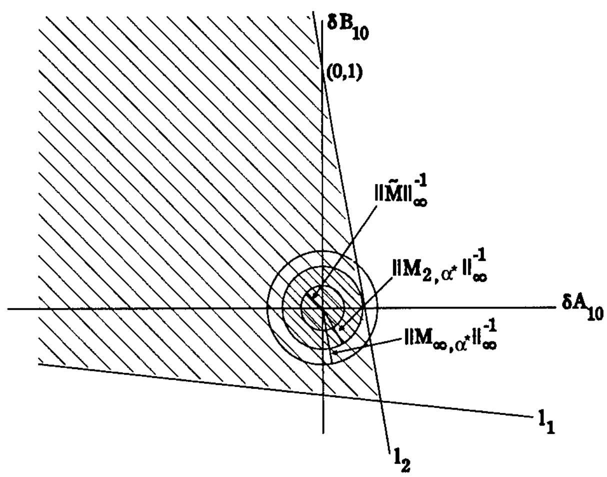

Figure 7.2: Allowable parameter space for the suboptimal H_2 design.

8. Conclusion

A promising control design methodology for trade-off between performance and stability margin for multivariable systems has been introduced and applied to a scalar example. To attain an analytical solution to this fundamental trade-off problem, a quadratic measure for stability margin is chosen. Furthermore, a novel approach is introduced which allows structured perturbations in the coprime polynomial matrix fraction description of the plant transfer matrix to be taken into account. The advocated methodology provides a good initial solution from which better solutions, if needed, may be searched for iteratively. Future work will include applications of this design methodology to practical multivariable examples to assess the efficacy of the methodology presented here. Needed in this effort are efficient numerical algorithms. In this regard, it is expected that [48–51] will play an important role.

Appendix A

The cost functional (4.1) is of the form

$$\hat{J} = J + \alpha^2 K \tag{8.1}$$

with

$$K(Z_{wz}) = \Delta E_{wz}. \tag{8.2}$$

The optimum Z_{wz} is the one which minimizes $\hat{J}$ and it depends parametrically on α. In this regard, one has the following lemma.

Lemma A1: J is a monotonically increasing function of α^2 and K is a monotonically decreasing function of α^2.

Proof: Suppose with $\alpha^2 = \alpha_1^2, Z_{wz} = Z_{wz_1}$ is the unique optimum choice and for this choice $J = J_1$ and $K = K_1$. Similarly, with $\alpha^2 = \alpha_2^2, Z_{wz} = Z_{wz_2}$ minimizes $\hat{J}$ and $J = J_2$ and $K = K_2$ in this case. Then,

$$J_1 + {\alpha_2}^2 K_1 > J_2 + {\alpha_2}^2 K_2 \tag{8.3}$$

and

$$J_2 + {\alpha_1}^2 K_2 > J_1 + {\alpha_1}^2 K_1. \tag{8.4}$$

Multiplying (8.3) with α_1^2 and (8.4) with ${\alpha_2}^2$ and adding the two expressions yields

$${\alpha_1}^2(J_1 - J_2) > {\alpha_2}^2(J_1 - J_2). \tag{8.5}$$

Therefore,

$$(J_1 - J_2)({\alpha_1}^2 - {\alpha_2}^2) > 0. \tag{8.6}$$

This implies that

$$(J_1 - J_2) < 0 \Leftrightarrow {\alpha_1}^2 - {\alpha_2}^2 < 0 \qquad (or\ \ (J_1 - J_2) > 0 \Leftrightarrow {\alpha_1}^2 - {\alpha_2}^2 > 0). \tag{8.7}$$

Therefore,

$$J_1 < J_2 \Leftrightarrow {\alpha_1}^2 < {\alpha_2}^2 \qquad (or\;\; J_1 > J_2 \Leftrightarrow {\alpha_1}^2 > {\alpha_2}^2). \tag{8.8}$$

The above establishes that J is a monotonically increasing function of α^2. From (8.3),

$$J_1 - J_2 > {\alpha_2}^2(K_2 - K_1). \tag{8.9}$$

Furthermore, from (8.7) and (8.9),

$${\alpha_1}^2 < {\alpha_2}^2 \Leftrightarrow 0 > J_1 - J_2 > {\alpha_2}^2(K_2 - K_1). \tag{8.10}$$

So,

$${\alpha_1}^2 < {\alpha_2}^2 \quad \Leftrightarrow \quad J_1 < J_2 \quad and \quad K_1 > K_2. \tag{8.11}$$

This completes the proof.

Appendix B

The proof of the theorem stated in Section IV is given in this appendix. The first step is to recall that minimizing (4.25) with respect to Z_{wz} is equivalent to doing the same to (4.21). This optimization problem is a special case of the one solved in [31]. Since R_1 and R_2 are strictly proper here and since it is clear from (4.20) with $\lambda_2 \neq 0$ that both $\sqrt{\lambda_2}U_2$ and V_2 have full column rank everywhere in the complex plane (infinity included), it follows from Theorem 3.4 in [31] that one can always write the optimal choice for Z_{wz}, which plays the role of Q in [31], as

$$Z^o_{wz} = Z_{wz_\infty} + \hat{Z}_{wz} \tag{8.12}$$

where the constant matrix $Z_{wz_\infty} = 0$ and $||\hat{Z}_{wz}||_2$ exists.

Now all that remains is the implementation of the steps given in Section IV of [31] to derive the formula for

$$Z^o_{wz} = \hat{Z}_{wz}. \tag{8.13}$$

First set

$$r = \begin{bmatrix} \mathrm{vec}(\sqrt{\lambda_1}R_1) \\ \mathrm{vec}(\sqrt{\lambda_2}R_2) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(\sqrt{\lambda_1}W_1\Psi_1H_2) \\ 0 \end{bmatrix} \tag{8.14}$$

and

$$W = \begin{bmatrix} V_1' \otimes (\sqrt{\lambda_1} U_1) \\ V_2' \otimes (\sqrt{\lambda_2} U_2) \end{bmatrix} = \begin{bmatrix} -H_2' \Psi_3' \otimes \sqrt{\lambda_1} W_1 \Psi_2 \\ -\sqrt{\lambda_2} I \end{bmatrix}. \tag{8.15}$$

Next recognize that $W_* = \tilde{W}$ in [31] and form

$$\tilde{W} W = W_* W = \lambda_1 \mathcal{D} \tag{8.16}$$

where $\mathcal{D}$ is given by (4.28) with

$$\alpha^2 = \frac{\lambda_2}{\lambda_1} \neq 0. \tag{8.17}$$

The matrix $W_* W$ has a Wiener spectral factor W_o. Hence, so does $\mathcal{D}$ and the symbol ∇ is used to represent it. Clearly, one can take

$$W_o = \sqrt{\lambda_1} \nabla. \tag{8.18}$$

Associated with this factorization is the inner matrix

$$W_i = W W_o^{-1} = \frac{1}{\sqrt{\lambda_1}} W \nabla^{-1}. \tag{8.19}$$

Using the fact that

$$P_{H_2}(\tilde{W_i} r) = \{W_{i*} r\}_+ \tag{8.20}$$

in (4.5) of [31] then gives

$$\mathrm{vec}(Z_{wz}^o) \triangleq z_o = W_o^{-1} \{W_{i*} r\}_+ \tag{8.21}$$

or

$$z_o = \frac{1}{\sqrt{\lambda_1}} \nabla^{-1} \left\{ \frac{1}{\sqrt{\lambda_1}} \nabla_*^{-1} W_* r \right\}_+ . \tag{8.22}$$

From (8.14) and (8.15), one gets

$$W_* r = -(H_2' \Psi_3' \otimes \sqrt{\lambda_1} W_1 \Psi_2)_* \ \mathrm{vec}(\sqrt{\lambda_1} W_1 \Psi_1 H_2). \tag{8.23}$$

The identities (see [52])

$$(G_1 \otimes G_2)_* = G_{1*} \otimes G_{2*} \tag{8.24}$$

and

$$\mathrm{vec}(G_1 G_2 G_3) = (G_3' \otimes G_1)\mathrm{vec}(G_2) \tag{8.25}$$

then lead to

$$W_* r = -\mathrm{vec}\left((\sqrt{\lambda_1}\Psi_{2*}W_{1*})(\sqrt{\lambda_1}W_1\Psi_1 H_2)(H_{2*}\Psi_{3*})\right) \tag{8.26}$$

or

$$W_* r = \lambda_1 v \tag{8.27}$$

where v is given by (4.29). Substituting (8.27) into (8.22) gives (4.27) and completes the proof.

9. References

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10. Publications and Technical Reports for this Grant

- Refereed Journal Papers (appeared, accepted, and submitted)

[1] J. J. Bongiorno, F. Khorrami, and T. C. Lin, "A Wiener-Hopf approach to tradeoffs between stability margin and performance in two and three degree-of-freedom multivariable control systems." Submitted to the *International Journal of Control.*

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- Refereed Conference Papers (appeared, accepted, and submitted)

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- Hemant Melkote; attained the M.S. degree in the Electrical Engineering and is currently continuing for his Ph.D.
- T.C. Lin; attained the M.S. degree in the Electrical Engineering and finished the course work for his Ph.D. degree.
- Dr. Z. Retchkiman; a post-doctoral fellow.